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The existence of almost periodic solutions of functional - differential equations

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The existence of almost periodic solutions of ordinary differential equations has been discussed by several authors under the assumption that the system has a bounded solution which has some kind of stability properties. Deysach and Sell [2] discussed the existence of an almost periodic solution under the assumption that a periodic system has a bounded uniformly stable solution. Miller [6] and Seifert [7] considered an almost periodic system. Miller assumed that a bounded solution is totally stable, and Seifert assumed that a bounded solution is Σ-stable which is equivalent to the stability under disturbances introduced by Sell [9]. All of them used the theory of the dynamical system, and hence the uniqueness of solutions is assumed. Coppel [1] has shown that the result of Miller can be obtained by using the property of the asymptotic almost periodic function introduced by Fréchet without assuming the uniqueness of solutions. In this article, we shall show that all of the results obtained by applying the theory of the dynamical system can be obtained for functional-differential equations by using properties of the asymptotic almost periodic function. This article is based on the papers by Yoshizawa [10] and by Kato and Yoshizawa [5].

Let $f(t)$ be a continuous vector function defined on $a \leqslant t < \infty$. $f(t)$ is said to be asymptotically almost periodic if it is a sum of a continuous almost periodic function $p(t)$ and a continuous function $q(t)$ defined on $a \leqslant t < \infty$ which tends to zero as $t \to \infty$, that is,

$$f(t) = p(t) + q(t) \quad (1)$$

For an asymptotically almost periodic function $f(t)$, its decomposition (1) is unique and $f(t)$

is bounded and uniformly continuous on $a \leqslant t < \infty$. Moreover, $f(t)$ is asymptotically almost periodic if and only if for any sequence $\{\tau_k\}$ such that $\tau_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{\tau_{k_j}\}$ for which $f(t+\tau_{k_j})$ converges uniformly on $a \leqslant t < \infty$. For our purpose, the following property is important.

Lemma 1. Suppose that an asymptotically almost periodic function $f(t)$ is differentiable and that its derivative $f'(t)$ is also asymptotically almost periodic. Then the decomposition of $f'(t)$ is

$$f'(t) = p'(t) + q'(t), \tag{2}$$

where $p'(t)$ and $q'(t)$ are the derivatives of $p(t)$ and $q(t)$ in (1), respectively.

In this paper, we shall consider a system of almost periodic functional-differential equations. For a given $h \geqslant 0$, let C denote the space of continuous functions defined on $[-h, 0]$. For an $x \in R^n$, let $|x|$ be any norm. For a $\varphi \in C$, we define the norm $\|\varphi\|$ of φ by $\|\varphi\| = \sup\{|\varphi(\theta)|\,;\, -h \leqslant \theta \leqslant 0\}$. Moreover, we denote by C_a the set of φ such that $\|\varphi\| \leqslant a$. Letting $\dot{x}(t)$ be the right-hand derivative of $x(t)$, consider a system of functional-differential equations

$$\dot{x}(t) = F(t, x_t), \tag{3}$$

where x_t will denote the function $x(t+\theta)$, $-h \leqslant \theta \leqslant 0$, that is, $x_t \in C$.

Definition. Let $F(t, \varphi)$ be a continuous function defined on $R \times C_a$ with values in R^n, where $R = (-\infty, \infty)$. $F(t, \varphi)$ is said to be almost periodic in t uniformly for $\varphi \in C_a$, if for any $\epsilon > 0$ and any compact set S in C_a, there exists an $\ell(\epsilon, S) > 0$ such that every interval of length $\ell(\epsilon, S)$ contains a τ for which

$$|F(t+\tau, \varphi) - F(t, \varphi)| \leqslant \epsilon \quad \text{for all } t \in R \text{ and } \varphi \in S. \tag{4}$$

Lemma 2. If $F(t, \varphi)$ is almost periodic in t uniformly for $\varphi \in C_a$, for any real sequence $\{\tau_k\}$ there is a subsequence $\{\tau_{k_j}\}$ of $\{\tau_k\}$ and a continuous function $G(t, \varphi)$ such that

$$(5) \qquad F(t + \tau_{k_j}, \varphi) \to G(t, \varphi)$$

uniformly on $R \times S$ as $j \to \infty$, where S is any compact set in C_a. Moreover, $G(t, \varphi)$ is almost periodic in t uniformly for $\varphi \in C_a$.

Now let $F(t, \varphi)$ be almost periodic in t uniformly for $\varphi \in C_a$. We shall denote by $T(F)$ the function space consisting of all translates of F, that is $F^\tau \in T(F)$ where $F^\tau(t, \varphi) = F(t + \tau, \varphi)$, $\tau \in R$. Let $H(F)$ be the uniform closure of $T(F)$ in the sense of (5). $H(F)$ is called the hull of $F(t, \varphi)$.

We assume that $F(t, \varphi)$ in (3) is continuous on $R \times C_{B^*}$ and $F(t, \varphi)$ is almost periodic in t uniformly for $\varphi \in C_{B^*}$. We shall denote by I the interval $0 \leqslant t < \infty$.

Theorem 1. Suppose that the system (3) has a solution $\xi(t)$ defined on I such that $\|\xi_t\| \leqslant B^*$ for $t \geqslant 0$. If the solution $\xi(t)$ is asymptotically almost periodic, then the system (3) has an almost periodic solution $p(t)$.

Proof. Since $\xi(t)$ is asymptotically almost periodic, it has the decomposition $\xi(t) = p(t) + q(t)$, where $p(t)$ is almost periodic and $q(t) \to 0$ as $t \to \infty$. We can see easily that $|p(t)| \leqslant B^*$ for all $t \in R$. Moreover, we can show that $F(t, p_t)$ is almost periodic in t. Since $\xi(t)$ is a solution of (3), we have

$$(6) \qquad \dot{\xi}(t) = F(t, p_t) + F(t, \xi_t) - F(t, p_t) .$$

On the other hand, it is clear that $F(t, \xi_t) - F(t, p_t) \to 0$ as $t \to \infty$, because $\xi_t = p_t + q_t$ for $t \geqslant h$. Thus (6) shows that $\dot{\xi}(t)$ is asymptotically almost periodic, and hence it follows from Lemma 1 that

(7) $$\dot{p}(t) = F(t, p_t) \quad \text{for} \quad t \in R,$$

which shows that $p(t)$ is an almost periodic solution of (3). This completes the proof.

Thus, when an almost periodic system has an asymptotically almost periodic solution, we always can see the existence of an almost periodic solution. Here it is noticed that we do not require the uniqueness of solution.

Definition. Let S be a given compact set in C_{B^*} and $\xi(t)$ be a solution of (3) such that $\xi_t \in S$ for all $t \geq 0$. For $G \in H(F)$ and $P \in H(F)$, define $\rho(G, P; S)$ by

$$\rho(G, P; S) = \sup \{ | G(t, \varphi) - P(t, \varphi) | ; \; t \in R, \; \varphi \in S \} .$$

The solution $\xi(t)$ is said to be stable under disturbances from $H(F)$ with respect to S for $t \geq 0$, if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\| \xi_{t+\tau} - x_t(0, \psi, G) \| \leq \epsilon$ for $t \geq 0$, whenever $G \in H(F)$, $\| \xi_\tau - \psi \| \leq \delta(\epsilon)$ and $\rho(F^\tau, G; S) \leq \delta(\epsilon)$ for some $\tau \geq 0$, where $x(0, \psi, G)$ is a solution of

(8) $$\dot{x}(t) = G(t, x_t)$$

such that $x_0(0, \psi, G) = \psi$ and $x_t(0, \psi, G) \in S$ for all $t \geq 0$.

We shall discuss the existence of an asymptotically almost periodic solution of the system (3).

Theorem 2. Let S be a compact set in C_{B^*}. If the system (3) has a solution $\xi(t)$ such that $\xi_t \in S$ for all $t \geq 0$ and if $\xi(t)$ is stable under disturbances from $H(F)$ with respect to S, then $\xi(t)$ is an asymptotically almost periodic solution of (3), and consequently the system (3) has an almost periodic solution.

Proof. Let $\{\tau_k\}$ be any sequence such that $\tau_k > 0$ and $\tau_k \to \infty$ as $k \to \infty$. Set $\xi^k(t) = \xi(t + \tau_k)$. Then $\xi^k(t)$ is a solution of

(9) $$\dot{x}(t) = F(t+\tau_k, x_t)$$

through $(0, \xi_{\tau_k})$ and clearly $\xi^k{}_t \in S$ for all $t \geq 0$. Moreover, it is clear that $\xi^k(t)$ is stable under disturbances from $H(F^{\tau_k})$ with respect to S with the same pair $(\epsilon, \delta(\epsilon))$ as the one for $\xi(t)$. Since $F(t, \varphi)$ is almost periodic in t uniformly for $\varphi \in C_{B^*}$ and S is a compact set in C_{B^*}, $\{\tau_k\}$ has a subsequence, which we shall denote by $\{\tau_k\}$ again, such that $F(t+\tau_k, \varphi)$ converges uniformly on RXS as $k \to \infty$, and hence there is an integer $k_0(\epsilon) > 0$ such that if $m \geq k \geq k_0(\epsilon)$,

(10) $$|F(t+\tau_k, \varphi) - F(t+\tau_m, \varphi)| \leq \delta(\epsilon) \quad \text{on } R \times S,$$

where $\delta(\epsilon)$ is the one for stability under disturbances. Therefore, if $m \geq k \geq k_0(\epsilon)$, we have $\rho(F^{\tau_k}, F^{\tau_m}; S) < \delta(\epsilon)$. Moreover, since $\xi^k{}_0 \in S$, we can assume that if $m \geq k \geq k_0(\epsilon)$,

(11) $$\| \xi^k{}_0 - \xi^m{}_0 \| \leq \delta(\epsilon),$$

taking a subsequence again, if necessary. Since $\xi^m(t)$ is a solution of

(12) $$\dot{x}(t) = F(t+\tau_m, x_t)$$

such that $\xi^m{}_t \in S$ for all $t \geq 0$, $F^{\tau_m} \in H(F^{\tau_k})$ and $\xi^k(t)$ is stable under disturbances from $H(F^{\tau_k})$ with respect to S, we have $\| \xi^k{}_t - \xi^m{}_t \| \leq \epsilon$ for $t \geq 0$ if $m \geq k \geq k_0(\epsilon)$. This implies that $|\xi(t+\tau_k) - \xi(t+\tau_m)| \leq \epsilon$ for all $t \geq 0$ if $m \geq k \geq k_0(\epsilon)$. This proves that $\xi(t+\tau_k)$ is uniformly convergent on I as $k \to \infty$. Thus $\xi(t)$ is asymptotically almost periodic. The existence of an almost periodic solution follows immediately from Theorem 1.

Now we consider the case where there exists an $L > 0$ such that $|F(t, \varphi)| \leq L$ on RXC_{B^*}. Let K denote the space of functions $\varphi(\theta)$ defined on $[-h, 0]$ such that $\|\varphi\| \leq B^*$ and $|\varphi(\theta) - \varphi(\theta')| \leq L|\theta - \theta'|$, $-h \leq \theta, \theta' \leq 0$. Then, clearly K is a compact set in C_{B^*}.

First of all, we shall consider the case where $F(t, \varphi)$ in (3) is periodic in t, that is, we assume that $F(t+\omega, \varphi) = F(t, \varphi)$, $\omega > 0$, on RXC_{B^*}. In the case where F is not autonomous on RXK, there is a smallest positive period ω^* of $F(t, \varphi)$ on RXK and we can see that for any $G \in H(F)$ and any $\tau \geqslant 0$ there is a $\sigma(\tau, G, K)$ such that $\tau - \frac{\omega^*}{2} \leqslant \sigma(\tau, G, K) \leqslant \tau + \frac{\omega^*}{2}$ and $G(t, \varphi) = F(t+\sigma, \varphi)$ on RXK. For such a $\sigma(\tau, G, K)$, we have the following lemma.

Lemma 3. For any $\epsilon > 0$, there exists a $\gamma(\epsilon) > 0$ such that if $\tau \geqslant 0$, $G \in H(F)$ and $\rho(F^{\tau}, G; K) \leqslant \gamma(\epsilon)$, then $|\tau - \sigma(\tau, G, K)| < \epsilon$.

Proof. Suppose that there is no $\gamma(\epsilon)$. Then, for some $\epsilon > 0$, there exist sequences $\{\gamma_k\}$, $\{\tau_k\}$ and $\{\sigma_k\}$ such that $\gamma_k \to 0$ as $k \to \infty$, $\tau_k \geqslant 0$, $\sup\{|F(t+\tau_k, \varphi) - F(t+\sigma_k, \varphi)|;\ t \in R,\ \varphi \in K\} < \gamma_k$, $\tau_k - \frac{\omega^*}{2} \leqslant \sigma_k \leqslant \tau_k + \frac{\omega^*}{2}$ and $|\tau_k - \sigma_k| \geqslant \epsilon$. Set $\tau_k = N_k \omega^* + \tau_k'$, where N_k is a nonnegative integer and $0 \leqslant \tau_k' < \omega^*$. If we set $\sigma_k = N_k \omega^* + \sigma_k'$, then $\tau_k' - \frac{\omega^*}{2} \leqslant \sigma_k' \leqslant \tau_k' + \frac{\omega^*}{2}$. Since $0 \leqslant \tau_k' < \omega^*$ and $-\frac{\omega^*}{2} \leqslant \sigma_k' \leqslant \omega^* + \frac{\omega^*}{2}$, there exist τ' and σ' such that $\tau_k' \to \tau'$, $\sigma_k' \to \sigma'$ as $k \to \infty$, taking a subsequence, if necessary. Then, $\tau' - \frac{\omega^*}{2} \leqslant \sigma' \leqslant \tau' + \frac{\omega^*}{2}$, that is, we have

$$(13) \qquad |\tau' - \sigma'| \leqslant \frac{\omega^*}{2}.$$

On the other hand, $\sup\{|F(t+\tau_k, \varphi) - F(t+\sigma_k, \varphi)|;\ t \in R,\ \varphi \in K\} = \sup\{|F(t+\tau_k', \varphi) - F(t+\sigma_k', \varphi)|;\ t \in R,\ \varphi \in K\} < \gamma_k$ and $\gamma_k \to 0$ as $k \to \infty$, and hence $F(t+\tau', \varphi) = F(t+\sigma', \varphi)$ on RXK. This shows that $|\tau' - \sigma'|$ is a period of $F(t, \varphi)$ on RXK. Since $|\tau_k' - \sigma_k'| = |\tau_k - \sigma_k| \geqslant \epsilon$ implies $\epsilon \leqslant |\tau' - \sigma'|$ and we have (13), this contradicts that ω^* is the smallest positive period on RXK. This proves the lemma.

Theorem 3. Let $\xi(t)$ be a solution of the periodic system (3) such that $|\xi(t)| \leqslant B$,

$B < B^*$, for all $t \geq 0$. If $\xi(t)$ is uniformly stable for $t \geq 0$, then $\xi(t)$ is stable under disturbances from $H(F)$ with respect to K for $t \geq h$.

Proof. Since $|\xi(t)| \leq B$ and $|\dot{\xi}(t)| \leq L$ for $t \geq 0$, clearly $\xi_t \in K$ for all $t \geq h$, and for a solution $x(h, \psi, G)$ of (8) through (h, ψ), $\psi \in K$, $x_t(h, \psi, G) \in K$ as long as $x(h, \psi, G)$ exists. If F is autonomous on K, that is, $F(t, \varphi) = F^*(\varphi)$ on K, then for any $G \in H(F)$, $G(t, \varphi) = F^*(\varphi)$ on K. Therefore, $\rho(F^\tau, G ; K) = 0$. Thus, since $\xi(t)$ is uniformly stable for $t \geq h$, it is clear that $\xi(t)$ is stable under disturbances from $H(F)$ for $t \geq h$.

Now we shall consider the case where F is not autonomous on K and we assume that ω^* is the smallest positive period of $F(t, \varphi)$ on $R \times K$. Since $\xi_t \in K$ for all $t \geq h$, we have $|\xi(t) - \xi(t')| < \frac{\delta(\epsilon)}{2}$ for $t, t' \in [0, \infty)$ if $|t - t'| < \frac{\delta(\epsilon)}{2L}$, where $\delta(\epsilon)$ is the one for uniform stability of $\xi(t)$. By Lemma 3, there is a $\gamma(\epsilon) > 0$ such that if $\tau \geq 0$, $G \in H(F)$ and $\rho(F^\tau, G ; K) \leq \gamma(\epsilon)$, then

$$|\tau - \sigma(\tau, G, K)| < \frac{\delta(\epsilon)}{2L},$$

where we can assume that $\gamma(\epsilon) < \frac{\delta(\epsilon)}{2}$ and $0 < \epsilon < \frac{B^* - B}{2}$. Moreover,

$$G(t, \varphi) = F(t + \sigma, \varphi) \quad \text{on} \quad R \times K .$$

For a fixed $\tau \geq 0$, let $\zeta(t) = \xi(t + \tau)$. Then $\zeta(t)$ is a solution through $(h, \xi_{h+\tau})$ of

$$\dot{x}(t) = F(t + \tau, x_t) . \tag{14}$$

Letting $\| \xi_{h+\tau} - \psi \| \leq \gamma(\epsilon)$, $\psi \in K$ and $G \in H(F)$ such that $\rho(F^\tau, G ; K) \leq \gamma(\epsilon)$, consider a solution $x(t)$ through (h, ψ) of (8). As long as $x(t)$ exists, $x_t \in K$, and hence $x(t)$ is a solution of

$$\dot{x}(t) = F(t + \sigma, x_t), \quad \sigma = \sigma(\tau, G, K), \tag{15}$$

throuth (h, ψ). Hence we have

(16) $$|\tau - \sigma| < \frac{\delta(\epsilon)}{2L} .$$

If we set $y(t) = \zeta(t + \sigma - \tau)$, then $y(t) = \xi(t + \sigma)$. First of all, we assume that $\sigma \geq 0$. Then $y(t)$ is a solution of (15) through $(0, \xi_\sigma)$ and $y(t)$ is uniformly stable for $t \geq 0$ with the same pair $(\epsilon, \delta(\epsilon))$ as the one for $\xi(t)$. Since we have (16), $|\xi(h + \sigma + \theta) - \xi(h + \tau + \theta)| < \frac{\delta(\epsilon)}{2}$ for $-h \leq \theta \leq 0$, and hence $\|\xi_{h+\sigma} - \xi_{h+\tau}\| < \frac{\delta(\epsilon)}{2}$. Since we have

$$\begin{aligned} \|y_h - \psi\| = \|\xi_{h+\sigma} - \psi\| &\leq \|\xi_{h+\sigma} - \xi_{h+\tau}\| + \|\xi_{h+\tau} - \psi\| \\ &< \frac{\delta(\epsilon)}{2} + \gamma(\epsilon) < \delta(\epsilon), \end{aligned}$$

the uniform stability of $y(t)$ implies that

(17) $$\|y_t - x_t\| < \epsilon \qquad \text{for } t \geq h .$$

Moreover, $|y(t + \theta) - \zeta(t + \theta)| = |\zeta(t + \sigma - \tau + \theta) - \zeta(t + \theta)| < \frac{\delta(\epsilon)}{2}$ for $t \geq h$, because of (16), and hence

(18) $$\|y_t - \zeta_t\| < \frac{\delta(\epsilon)}{2} < \epsilon \qquad \text{for } t \geq h .$$

From (17) and (18), it follows that $\|\zeta_t - x_t\| < 2\epsilon$ for all $t \geq h$, or

$$\|\xi_{t+\tau} - x_t\| < 2\epsilon \qquad \text{for all } t \geq h .$$

Next we shall consider the case where $\sigma < 0$, and consequently, $\tau - \sigma > 0$. If we set $z(t) = x(t + \tau - \sigma)$, then $z(t)$ is a solution of (14) through $(h, x_{h+\tau-\sigma})$, and $z_t \in K$ for all $t \geq h$. Since (16) implies that $\|\psi - x_{h+\tau-\sigma}\| < \frac{\delta(\epsilon)}{2}$, we have

$$\|\zeta_h - z_h\| = \|\xi_{h+\tau} - x_{h+\tau-\sigma}\| \leq \|\xi_{h+\tau} - \psi\| + \|\psi - x_{h+\tau-\sigma}\| < \delta(\epsilon) .$$

Thus we have $\| \zeta_t - z_t \| < \epsilon$ for all $t \geqslant h$, because $\zeta(t)$ is uniformly stable. Moreover, $| z(t+\theta) - x(t+\theta) | = | x(t+\tau-\sigma+\theta) - x(t+\theta) | < \frac{\delta(\epsilon)}{2} < \epsilon$ for all $t \geqslant h$ and $-h \leqslant \theta \leqslant 0$, and hence $\| z_t - x_t \| < \epsilon$ for all $t \geqslant h$. Thus we have $\| \zeta_t - x_t \| < 2\epsilon$ for all $t \geqslant h$, or $\| \xi_{t+\tau} - x_t \| < 2\epsilon$ for all $t \geqslant h$. Thus we see that $\xi(t)$ is stable under disturbances from $H(F)$ with respect to K for $t \geqslant h$.

Corollary. Let $\xi(t)$ be a solution of the periodic system (3) such that $| \xi(t) | \leqslant B$, $B < B^*$, for all $t \geqslant 0$. If $\xi(t)$ is uniformly stable for $t \geqslant 0$, then $\xi(t)$ is asymptotically almost periodic, and consequently the periodic system (3) has an almost periodic solution.

Remark. Under the condition in Corollary, actually the system (3) has a uniformly stable almost periodic solution [10].

Theorem 4. Let $\xi(t)$ be a solution of the periodic system (3) such that $| \xi(t) | \leqslant B$, $B < B^*$, for all $t \geqslant 0$. If $\xi(t)$ is uniformly asymptotically stable for $t \geqslant 0$, then the periodic system (3) has a periodic solution of period $m\omega$ for some integer $m \geqslant 1$.

Proof. Set $\xi^k(t) = \xi(t + k\omega)$. By the above corollary, $\xi(t)$ is asymptotically almost periodic, and hence a subsequence $\{ \xi^{k_j}(t) \}$ converges uniformly on I as $j \to \infty$. Since $\xi^{k_j}{}_0$ is convergent, there is an integer k_p such that $\| \xi^{k_p}{}_0 - \xi^{k_{p+1}}{}_0 \| < \delta_0$, where δ_0 is the one for uniformly asymptotic stability of $\xi(t)$. Set $m = k_{p+1} - k_p$ and consider the solution $\xi(t + m\omega)$ of (3). Then we have

$$\| \xi^m{}_{k_p\omega} - \xi_{k_p\omega} \| \doteq \| \xi_{k_{p+1}\omega} - \xi_{k_p\omega} \| = \| \xi^{k_{p+1}}{}_0 - \xi^{k_p}{}_0 \| < \delta_0,$$

and hence

$$\| \xi^m{}_t - \xi_t \| \to 0 \quad \text{as } t \to \infty. \tag{19}$$

On the other hand, $\xi(t) = p(t) + q(t)$, where $p(t)$ is almost periodic and $q(t) \to 0$ as $t \to \infty$. Therefore, by (19), $|p(t) - p(t + m\omega)| \to 0$ as $t \to \infty$, which implies that $p(t) = p(t + m\omega)$ for all $t \in R$. This shows that the system (3) has a periodic solution $p(t)$ of period $m\omega$, because $p(t)$ is a solution of (3).

Remark. If $\xi(t)$ is uniformly asymptotically stable in the large, we have $\| \xi_{t+\omega} - \xi_t \| \to 0$ as $t \to \infty$, and hence $p(t) = p(t + \omega)$.

Remark. Under the condition in Theorem 4, actually the system (3) has a periodic solution which is uniformly asymptotically stable. Sell [8] proved this theorem for ordinary differential equations by considering dynamical systems, and Halanay [3, 4] proved the existence of a periodic solution of period $m\omega$ under a weaker condition on $\xi(t)$.

Now we shall consider the system (3) in which $F(t, \varphi)$ is almost periodic in t uniformly for $\varphi \in C_{B^*}$. Coppel [1] and Miller [6] have obtained an existence theorem for an almost periodic solution of ordinary differential equation under the assumption that a bounded solution is totally stable.

Definition. Let $\xi(t)$ be a solution of system (3) which satisfies $\| \xi_t \| \leq B$, $B < B^*$, for all $t \geq 0$. The solution $\xi(t)$ is said to be totally stable for $t \geq 0$, if for any $t_0 \geq 0$ and any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $g(t)$ is continuous on $[t_0, \infty)$ and satisfies $|g(t)| < \delta(\epsilon)$ for all $t \geq t_0$ and if $\psi \in C_{B^*}$ satisfies $\| \psi - \xi_{t_0} \| < \delta(\epsilon)$, then any solution $y(t)$ through (t_0, ψ) of the system

$$\dot{x}(t) = F(t, x_t) + g(t) \tag{20}$$

satisfies $\| \xi_t - y_t \| < \epsilon$ for all $t \geq t_0$.

Theorem 5. Let $\xi(t)$ be a solution of the system (3) which satisfies $\| \xi_t \| \leq B$, $B < B^*$, for all $t \geq 0$. If $\xi(t)$ is totally stable for $t \geq 0$, then it is stable under disturbances from $H(F)$ with respect to K for $t \geq h$.

Proof. For $G \in H(F)$ and $P \in H(F)$, $\rho(G, P; K) = \sup \{ | G(t, \varphi) - P(t, \varphi) | ; t \in R, \varphi \in K \}$, and hence $\rho(F^\tau, G; K) \leq \delta'(\epsilon) < \delta(\epsilon)$ implies that

$$(21) \qquad | F(t+\tau, \varphi) - G(t, \varphi) | < \delta(\epsilon) \quad \text{on} \quad R \times K,$$

where $\delta(\epsilon)$ is the one for total stability of $\xi(t)$. Let ψ satisfy $\| \xi_{h+\tau} - \psi \| \leq \delta'(\epsilon)$ and $\psi \in K$. Clearly $\xi_t \in K$ for all $t \geq h$. Let $\zeta(t)$ be a solution of (8) through (h, ψ). Then, $y(t) = \zeta(t-\tau)$, $t \geq h+\tau$, is a solution of $\dot{x}(t) = G(t-\tau, x_t)$ through $(h+\tau, \psi)$. In other words, $y(t)$ is a solution of

$$(22) \qquad \dot{x}(t) = F(t, x_t) + G(t-\tau, y_t) - F(t, y_t)$$

such that $y_{h+\tau} = \psi$. Clearly, $y_t \in K$ for $t \geq h+\tau$, and hence, as long as $y(t)$ exists,

$$(23) \qquad | G(t-\tau, y_t) - F(t, y_t) | < \delta(\epsilon)$$

by (21). Since $\xi(t)$ is a totally stable solution of (3) and $\| \xi_{h+\tau} - \psi \| < \delta(\epsilon)$ and we have (23), $\| y_t - \xi_t \| < \epsilon$ as long as $y(t)$ exists, which implies that $y(t)$ exists for all $t \geq h+\tau$ and $\| y_t - \xi_t \| < \epsilon$ for $t \geq h+\tau$. Replacing t by $t+\tau$, we have $\| \zeta_t - \xi_{t+\tau} \| < \epsilon$ for all $t \geq h$. This proves the theorem.

Corollary. Under the assumptions of Theorem 5, $\xi(t)$ is an asymptotically almost periodic solution of the system (3), and consequently the system (3) has an almost periodic solution.

Now we shall discuss the relationship between the uniformly asymptotic stability and total stability. For the almost periodic system (3), in addition to the assumption that $|F(t, \varphi)| \leq L$ for a constant $L > 0$ and all $(t, \varphi) \in R \times C_{B^*}$, we assume that for every $G \in H(F)$ the solution of (8) is unique for the initial condition.

Lemma 4. Under the assumptions above, let $T>0$, $B_1(<B^*)$ and K^* be given, where K^* is a compact subset of C_{B^*}. Then, for any $\epsilon>0$ there exists a $\delta(\epsilon)>0$ such that for any $t_0 \in I$, if $x(t)$ is a solution of the system (3) which satisfies $x_{t_0} \in K^*$ and $\|x_t\| \leqslant B_1$ for all $t_0 \leqslant t \leqslant t_0+T$, and if $g(t)$ is a continuous function such that $|g(t)|<\delta(\epsilon)$ on $[t_0, t_0+T]$, we have

$$\|x_t - y_t\| < \epsilon \qquad \text{for all} \quad t \in [t_0, t_0+T],$$

whenever $y(t)$ is a solution of the system (20) satisfying $\|x_{t_0} - y_{t_0}\| < \delta(\epsilon)$.

For the proof of this lemma, see [5].

Theorem 6. Under the assumptions above, we assume that the system (3) has a solution $\xi(t)$ defined on $[0,\infty)$ which satisfies $\|\xi_t\| \leqslant B$ for all $t \geqslant 0$ and some $B<B^*$ and is uniformly asymptotically stable for $t \geqslant 0$. Then $\xi(t)$ is totally stable for $t \geqslant 0$.

Proof. Since $\xi(t)$ is uniformly asymptotically stable for $t \geqslant 0$, for any $\epsilon>0$ there exists a $\delta_1(\epsilon)>0$ and a $T(\epsilon)>0$ such that if $x(t)$ is a solution of (3) satisfying $\|x_{t_1} - \xi_{t_1}\| < \delta_1(\epsilon)$ at some $t_1 \geqslant 0$, then $\|x_t - \xi_t\| < \frac{\epsilon}{2}$ for all $t \geqslant t_1$ and $\|x_t - \xi_t\| < \frac{\delta_1(\epsilon)}{2}$ for all $t \geqslant t_1 + T(\epsilon)$. Here we can assume that $T(\epsilon) > h$ and $\epsilon < B^* - B$. Since the initial function of $\xi(t)$ is continuous on $[-h, 0]$ and $\xi(t)$ satisfies $|\xi(t)| \leqslant B$ for all $t \geqslant 0$ and $|\dot{\xi}(t)| \leqslant L$ for all $t \geqslant 0$, $\{\xi_t;\ t \geqslant 0\}$ is contained in a compact subset K_1 of C_{B^*}. Set $K^* = K_1 \cup K_2$, where K_2 is the compact subset of C_{B^*} which is the set of $\varphi \in C_{B^*}$ such that $|\varphi(\theta) - \varphi(\theta')| \leqslant L^*|\theta - \theta'|$ for all $\theta, \theta' \in [-h, 0]$ and for a fixed constant $L^* > L$. For K^*, $B_1 = \frac{B^*+B}{2}$, $\frac{\delta_1(\epsilon)}{2}$ and $T(\epsilon)$, by Lemma 4, there exists a $\delta(\epsilon)>0$, $\delta(\epsilon) \leqslant \min\{\frac{\delta_1(\epsilon)}{2}, L^* - L\}$, such that for any $s \geqslant 0$, if $x(t)$ is a solution of (3) defined on $s \leqslant t \leqslant s + T(\epsilon)$ which satisfies $x_s \in K^*$ and $\|x_t\| \leqslant B_1$ and if $\|x_s - \varphi\| < \delta(\epsilon)$ and

$|g(t)| < \delta(\epsilon)$, then a solution $y(t)$ through (s, φ) of the system (20) exists on $[s, s+T(\epsilon)]$ and satisfies

$$\|x_t - y_t\| < \frac{\delta_1(\epsilon)}{2} \qquad \text{on } [s, s+T(\epsilon)].$$

For a fixed $t_0 \geqslant 0$, consider a system (20), where $|g(t)| < \delta(\epsilon)$ for all $t \geqslant t_0$, and a solution $y(t)$ of (20) such that $\|\xi_{t_0} - y_{t_0}\| < \delta(\epsilon)$. Since $\xi_{t_0} \in K^*$ and $\delta_1(\epsilon) < \epsilon$, we have

$$\|\xi_t - y_t\| < \frac{\delta_1(\epsilon)}{2} < \epsilon \qquad \text{on } [t_0, t_0 + T(\epsilon)] .$$

Since $\|\xi_{t_0+T(\epsilon)} - y_{t_0+T(\epsilon)}\| < \frac{\delta_1(\epsilon)}{2}$, $T(\epsilon) > h$ and $\delta_1(\epsilon) < B^* - B$, we have $\|y_{t_0+T(\epsilon)}\| \leqslant B_1$ and $|\dot{y}(t)| \leqslant L + \delta(\epsilon) \leqslant L^*$ on $t_0 + T(\epsilon) - h \leqslant t \leqslant t_0 + T(\epsilon)$. Therefore $y_{t_0+T(\epsilon)} \in K_2 \subset K^*$. Let $x(t)$ be a solution of (3) through $(t_0 + T(\epsilon), y_{t_0+T(\epsilon)})$. Since $\|\xi_{t_0+T(\epsilon)} - x_{t_0+T(\epsilon)}\| = \|\xi_{t_0+T(\epsilon)} - y_{t_0+T(\epsilon)}\| < \delta_1(\epsilon)$, we have

$$\|\xi_t - x_t\| < \frac{\epsilon}{2} \qquad \text{for all } t \geqslant t_0 + T(\epsilon)$$

and

$$\|\xi_{t_0+2T(\epsilon)} - x_{t_0+2T(\epsilon)}\| < \frac{\delta_1(\epsilon)}{2} .$$

On the other hand, we have

$$\|x_t - y_t\| < \frac{\delta_1(\epsilon)}{2} \qquad \text{on } t_0 + T(\epsilon) \leqslant t \leqslant t_0 + 2T(\epsilon),$$

because $\|x_{t_0+T(\epsilon)} - y_{t_0+T(\epsilon)}\| = 0$, $x_{t_0+T(\epsilon)} \in K^*$, $|g(t)| < \delta(\epsilon)$ and $\|x_t\| \leqslant \|\xi_t\| + \frac{\epsilon}{2} \leqslant B_1$ for all $t \geqslant t_0 + T(\epsilon)$. Thus we have

$$\|\xi_t - y_t\| < \epsilon \qquad \text{on } t_0 + T(\epsilon) \leqslant t \leqslant t_0 + 2T(\epsilon)$$

and

$$\| \xi_{t_0+2T(\epsilon)} - y_{t_0+2T(\epsilon)} \| < \delta_1(\epsilon) .$$

By the same argument, in general, we have

$$\| \xi_t - y_t \| < \epsilon \qquad \text{on} \quad t_0 + pT(\epsilon) \leqslant t \leqslant t_0 + (p+1)T(\epsilon) , \quad p = 2,\ 3,\ \ldots\ ,$$

and hence we have $\| \xi_t - y_t \| < \epsilon$ for all $t \geqslant t_0$, whenever $t_0 \geqslant 0$, $\| \xi_{t_0} - y_{t_0} \| < \delta(\epsilon)$ and $| g(t) | < \delta(\epsilon)$ for all $t \geqslant t_0$. This proves that $\xi(t)$ is totally stable for $t \geqslant 0$.

By the corollary of Theorem 5, we have the following corollary.

Corollary. Under the assumptions in Theorem 6, $\xi(t)$ is asymptotically almost periodic, and consequently the system (3) has an almost periodic solution.

Remark. In this case, actually we have a uniformly asymptotically stable almost periodic solution [10] .

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